A NEW MULTIPLICITY FORMULA FOR THE WEYL MODULES OF TYPE A†

YE JIACHEN[1] ZHOU ZHONGGUO[1, 2]

[1]*Department of Applied Mathematics, Tongji University, Shanghai 200092, People's Republic of China*
E-Mail: jcye@mail.tongji.edu.cn

[2]*Department of Mathematics, Jinan University, Jinan Shandong 250022, People's Republic of China*
E-Mail: zhgzhou@21cn.com

Abstract: A monomial basis and a filtration of subalgebras for the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}_l$ of type A_l is given in this note. In particular, a new multiplicity formula for the Weyl module $V(\lambda)$ of $\mathfrak{U}(\mathfrak{g}_l)$ is obtained in this note.

Keywords: Simple Lie algebra, Multiplicity formula, Weight.

2000 MR Subject Classification: 17B10 20G05

Let $\mathfrak{g}_l$ be a complex simple Lie algebra of type A_l, and $\mathfrak{U} = \mathfrak{U}(\mathfrak{g}_l)$ its universal enveloping algebra. For any dominant integer weiget $\lambda \in \Lambda_+$, $V(\lambda)$ denotes a finite dimensional irreducible $\mathfrak{U}(\mathfrak{g}_l)$-module, the Weyl module. Following Littelmann [2], we define a new monomial basis and a filtration of subalgebras for $\mathfrak{U}(\mathfrak{g}_l)$. Furthermore, we obtain a new basis and a new multiplicity formula for the Weyl module $V(\lambda)$ of $\mathfrak{U}(\mathfrak{g}_l)$ in this note.

This paper is organized as follows: we introduce an ordering relation on $\mathfrak{U}(\mathfrak{g}_l)^-$ in the first section; we define a new basis of $\mathfrak{U}(\mathfrak{g}_l)^-$ in Section 2; we record some useful commutative formulas and construct a filtration of subalgebras of $\mathfrak{U}(\mathfrak{g}_l)$ in Section 3; our main results concerning a new basis and a new multiplicity formula for the Weyl module $V(\lambda)$ of $\mathfrak{U}(\mathfrak{g}_l)$ is given in Section 4; several examples for $\mathfrak{g}_l$ being of type A_2 and A_4 is given in the last section. We shall freely use the notations in [1] without further comments.

We believe that our method could be generalized to the case of D_l at least. Moreover, our results may also be generalized to the cases of B_l and C_l. We will deal with them in a further note.

† Supported in part by the NNSFC (10271088).

1. An ordering relation on $\mathfrak{U}(\mathfrak{g}_l)^-$

1.1. Let $\Delta=\{\alpha_1,\alpha_2,\cdots,\alpha_l\}$ be the set of simple roots. Set $\alpha_{i\ i}=\alpha_i$, and

$$\alpha_{i\ j}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j, \qquad \text{with} \qquad 1\le i\le j\le l.$$

Then

$$\Phi^+=\{\alpha_{i\ j}, \quad 1\le i<j\le l\}$$

is the set of positive roots which has $\frac{1}{2}l(l+1)$ elements. Fix an ordering of positive roots as follows:

$$\alpha_1,\alpha_2,\alpha_{1\ 2},\cdots,\alpha_i,\alpha_{i-1\ i},\cdots,\alpha_{1\ i},\cdots,\alpha_l,\alpha_{l-1\ l},\cdots,\alpha_{1\ l}.$$

Then define a Chevalley basis of $\mathfrak{g}_l$

$$e_i=e_{\alpha_i},e_{i\ j}=e_{\alpha_{i\ j}},f_i=f_{\alpha_i},f_{i\ j}=f_{\alpha_{i\ j}},h_i, \quad 1\le i\le l,\ \ 1\le i<j\le l,$$

accordingly. Let $\mathbb{N}$ be the set of non-negative integers. The Kostant $\mathbb{Z}$-form $\mathfrak{U}_{\mathbb{Z}}$ of $\mathfrak{U}$ is the $\mathbb{Z}$-subalgebra of $\mathfrak{U}$ generated by the elements $e_\alpha^{(k)}:=e_\alpha^k/k!$, $f_\alpha^{(k)}:=f_\alpha^k/k!$ for $\alpha\in\Phi^+$ and $k\in\mathbb{N}$. Set

$$\binom{h_i+c}{k}:=\frac{(h_i+c)(h_i+c-1)\cdots(h_i+c-k+1)}{k!}.$$

Then $\binom{h_i+c}{k}\in\mathfrak{U}_{\mathbb{Z}}$ for $1\le i\le l, c\in\mathbb{Z}, k\in\mathbb{N}$. Let $\mathfrak{U}^+,\mathfrak{U}^-,\mathfrak{U}^0$ be the positive part, negative part and zero part of $\mathfrak{U}$, respectively. They are generated by $e_\alpha^{(k)}$, $f_\alpha^{(k)}$ and $\binom{h_i}{k}$ with $k\in\mathbb{N},\alpha\in\Phi^+$ and $1\le i\le l$, respectively. The algebra $\mathfrak{U}$ is a Hopf algebra which has a triangular decomposition $\mathfrak{U}=\mathfrak{U}^-\mathfrak{U}^0\mathfrak{U}^+$. It is known that the PBW-type basis for $\mathfrak{U}$ has the form of

$$\prod_{\alpha\in R_+} f_\alpha^{(a_\alpha)}\prod_{i=1}^{l}\binom{h_i}{b_i}\prod_{\alpha\in R_+} e_\alpha^{(c_\alpha)}$$

with $a_\alpha,b_i,c_\alpha\in\mathbb{N}$. In particular, if we define

$$I=(i_1,i_2,\cdots,i_{\frac{l(l+1)}{2}})\in\mathbb{N}^{\frac{l(l+1)}{2}},$$

then

$$f^I=f_1^{(i_1)}f_2^{(i_2)}f_{1\ 2}^{(i_3)}\cdots f_{1\ l}^{(i_{\frac{l(l+1)}{2}})}$$

forms a PBW-type basis of $\mathfrak{U}^-$ with all $I\in\mathbb{N}^{\frac{l(l+1)}{2}}$. In particular, one has $f^{\mathbf{0}}=1$ when $I=(0,0,\cdots,0)=\mathbf{0}$.

1.2. First of all, we define an ordering on $\mathbb{N}^{\frac{l(l+1)}{2}}$ "$\prec$"as follows: for any $I,I'\in\mathbb{N}^{\frac{l(l+1)}{2}}$, $I=(i_1,i_2,\cdots,i_{\frac{l(l+1)}{2}})$ and $I'=(i'_1,i'_2,\cdots,i'_{\frac{l(l+1)}{2}})$, if there exists a k with $1\le k\le\frac{l(l+1)}{2}$ such that $i_k<i'_k$ and $i_j=i'_j$ for all $j>k$, then we say $I\prec I'$; otherwise, one has $I=I'$. It is easy to see that for any $I,I'\in\mathbb{N}^{\frac{l(l+1)}{2}}$, if $I\ne I'$, we must have either $I\prec I'$ or $I'\prec I$. Therefore, we can define an ordering on $\mathfrak{U}^-$ " $\prec$ " in the same

way: we say $f^I \prec f^{I'}$ if and only if $I \prec I'$. It is easy to see that different basis elements do not be equal. Any element in $\mathfrak{U}^-$ can be written uniquely in terms of

$$f = \sum_{I \in \mathbb{N}^{\frac{l(l+1)}{2}}} a_I f^I, \qquad \text{with} \qquad a_I \in \mathbb{C}.$$

Moreover, we can define the leading element of f $\max f = f^I$ when all the other $f^{I'} \prec f^I$ with $a_{I'} \neq 0$. Therefore, one has the following claim:

1.3. If $f_1, f_2, \cdots, f_m \in \mathfrak{U}^-$ with $\max f_1 \prec \max f_2 \prec \cdots \prec \max f_m$. Then f_1, f_2, $\cdots$, f_m are linearly independent.

2. Some commutator formulas and a class of special subalgebras in $\mathfrak{U}(\mathfrak{g}_l)$

2.1. For $1 \leq i, j \leq l$, one has the following commutator formulas (cf. [1]).

(1) $\quad e_i f_j = f_j e_i, \quad \text{when } i \neq j;$

(2) $$e_i^{(a)} f_i^{(b)} = \sum_{k=0}^{\min(a,b)} f_i^{(b-k)} \binom{h_i - a - b + 2k}{k} e_i^{(a-k)};$$

(3) $$h_i f_j^{(k)} = f_j^{(k)} h_i - k\alpha_j(h_i) f_j^{(k)};$$

(4) $$\binom{h_i + a}{b} f_j^{(k)} = f_j^{(k)} \binom{h_i - k\alpha_j(h_i) + a}{b};$$

(5) $$\begin{aligned} e_i f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_1^{(a_1)} &= f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_1^{(a_1)} e_i + \\ + f_1^{(a_l)} \cdots f_i^{(a_i-1)} \left(h_i - a_i + 1\right) f_{i-1}^{(a_{i-1})} \cdots f_1^{(a_1)} &= f_l^{(a_l)} \cdots f_i^{(a_i)} \cdots f_1^{(a_1)} e_i + \\ + f_1^{(a_l)} \cdots f_i^{(a_i-1)} \cdots f_1^{(a_1)} &\left(h_i - a_i + 1 - \sum_{k=1}^{i-1} a_k \alpha_k(h_i) \right). \end{aligned}$$

2.2. Furthermore, elements $f_i, f_{i\ j}$, $(1 \leq i < j \leq l)$ satisfy the following commutator relations (cf. [4] or [5]):

(1) $\quad f_{i+1} f_i = f_i f_{i+1} + f_{i\ i+1};$

(2) $\quad f_i f_j = f_j f_i, \quad \text{when } |i - j| \neq 1;$

(3) $\quad f_{i+1\ j} f_i = f_i f_{i+1\ j} + f_{i\ j} \quad \text{or} \quad f_{j+1} f_{i\ j} = f_{i\ j} f_{j+1} + f_{i\ j+1};$

(4) $\quad f_{i\ j} f_k = f_k f_{i\ j}, \quad \text{when } i - k \neq 1 \quad \text{or} \quad k - j \neq 1;$

(5) $\quad f_{j+1\ k} f_{i\ j} = f_{i\ j} f_{j+1\ k} + f_{i\ k};$

(6) $\quad f_{i\ j} f_{k\ h} = f_{k\ h} f_{i\ j}, \quad \text{when } k - j \neq 1 \quad \text{or} \quad i - h \neq 1;$

(7) $$f_{i+1}^{(a)} f_i^{(b)} = \sum_{k=0}^{\min(a,b)} f_{i\ i+1}^{(k)} f_i^{(b-k)} f_{i+1}^{(a-k)}, \quad 1 \leq i \leq l-1.$$

2.3. Let us construct a class of special subalgebras $\mathfrak{U}(\mathfrak{g}_i), 1 \leq i \leq l$, of $\mathfrak{U}$ as follows. Set

$$\mathfrak{U}(\mathfrak{g}_i) = \langle e_j^{(a_j)}, f_j^{(b_j)}, \binom{h_j + c}{k} \mid a_j, b_j, c, k \in \mathbb{N}, 1 \leq j \leq i \rangle.$$

Then one has

$$0 \subseteq \mathfrak{U}(\mathfrak{g}_1) \subseteq \mathfrak{U}(\mathfrak{g}_2) \subseteq \cdots \subseteq \mathfrak{U}(\mathfrak{g}_l) = \mathfrak{U}.$$

The set of positive roots in $\mathfrak{U}(\mathfrak{g}_i)$ is just that of the first $\frac{1}{2}i(i+1)$ roots according to the ordering of Φ^+.

3. A monomial basis of $\mathfrak{U}(\mathfrak{g}_l)^-$

3.1. Let $K = (k_1^l, k_2^{l-1}, k_1^{l-1}, \cdots, k_i^{l-i+1}, k_{i-1}^{l-i+1}, \cdots, k_1^{l-i+1}, \cdots, k_l^1, k_{l-1}^1, \cdots, k_1^1) \in \mathbb{N}^{\frac{l(l+1)}{2}}$. Define an index set

$$\Pi := \{K \in \mathbb{N}^{\frac{l(l+1)}{2}} \mid k_i^{l-i+1} \geq k_{i-1}^{l-i+1} \geq \cdots \geq k_1^{l-i+1},\ 1 \leq i \leq l\}.$$

For any $K \in \Pi$, one has such a monomial

$$\begin{aligned}\theta^K =& f_1^{(k_1^l)} f_2^{(k_2^{l-1})} f_1^{(k_1^{l-1})} \cdots f_i^{(k_i^{l-i+1})} f_{i-1}^{(k_{i-1}^{l-i+1})} \cdots f_1^{(k_1^{l-i+1})} \cdots \\ & f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} \in \mathfrak{U}^-.\end{aligned}$$

The following theorem was first proved by Littelmann [2].

3.2. Theorem *The set $\{\theta^K \mid K \in \Pi\}$ forms a basis of the $\mathbb{Z}$-form of $\mathfrak{U}^-$.*

Proof First of all, we have to show that elements of the set $\{\theta^K \mid K \in \Pi\}$ are linearly independent.

Since $\{f^I \mid I \in \mathbb{N}^{\frac{l(l+1)}{2}}\}$ forms a PBW-type basis of $\mathfrak{U}^-$, one has for any $K \in \Pi$, $\theta^K \in \mathfrak{U}^-$ and

$$\theta^K = \sum_I a_I f^I, \quad a_I \in \mathbb{Z},\ \ I \in \mathbb{N}^{\frac{l(l+1)}{2}}.$$

Moreover, for any $K \in \Pi$, one has $I(K) = (k_1^l, k_2^{l-1} - k_1^{l-1}, k_1^{l-1}, \cdots, k_i^{l-i+1} - k_{i-1}^{l-i+1}, k_{i-1}^{l-i+1} - k_{i-2}^{l-i+1}, \cdots, k_2^{l-i+1} - k_1^{l-i+1}, k_1^{l-i+1}, \cdots, k_l^1 - k_{l-1}^1, k_{l-1}^1 - k_{l-2}^1, \cdots, k_2^1 - k_1^1, k_1^1) \in \mathbb{N}^{\frac{l(l+1)}{2}}$, because $k_i^j \geq k_{i-1}^j$ for all $1 \leq i, j \leq l$ with $i + j \leq l+1$. It is easy to calculate that

$$\max \theta^K = f^{I(K)}, \quad \text{with coefficient } 1.$$

Therefore, one has

$$\theta^K = f^{I(K)} + \sum_{I \prec I(K)} a_I f^I.$$

Note the fact that various θ^Ks and $\max \theta^K$s are different, when the corresponding Ks are different. We can conclude that elements of the set $\{\theta^K \mid K \in \Pi\}$ are linearly independent.

Next we show that the set $\{\theta^K \mid K \in \Pi\}$ generate $\mathfrak{U}_{\mathbb{Z}}^-$. For any $I = (i_1^l, i_2^{l-1}, i_1^{l-1}, \cdots, i_i^{l-i+1}, i_{i-1}^{l-i+1}, \cdots, i_1^{l-i+1}, \cdots, i_l^1, i_{l-1}^1, \cdots, i_1^1) \in \mathbb{N}^{\frac{l(l+1)}{2}}$, we define $K(I) = (i_1^l, i_2^{l-1} + i_1^{l-1}, i_1^{l-1}, \cdots, \sum_{p=1}^{j} i_p^{l-j+1}, \cdots, i_2^{l-j+1} + i_1^{l-j+1}, i_1^{l-j+1}, \cdots, \sum_{p=1}^{l} i_p^1, \sum_{p=1}^{l-1} i_p^1, \cdots, i_2^1 + i_1^1, i_1^1) \in \Pi$. Then one has

$$\theta^{K(I)} = f^I + \sum_{I' \prec I} a_{I'} f^{I'}.$$

An easy induction on the ordering of $\mathbb{N}^{\frac{l(l+1)}{2}}$ shows that

$$f^I = \theta^{I(K)} + \sum_{K' \in \Pi} c_{_{K'}} \theta^{K'} \quad \text{with } \ c_{_{K'}} \in \mathbb{Z}.$$

Combining the above facts, we show that the set $\{\theta^K |\ K \in \Pi\}$ forms a basis of the $\mathbb{Z}$-form of $\mathfrak{U}^-$.

3.3. Define $\Pi_{l-1} := \{K \in \Pi |\ k_j^1 = 0,\ 1 \le j \le l\} \subseteq \Pi$. We can see from the above discussion that the set $\{\theta^K |\ K \in \Pi_{l-1}\}$ forms a basis of the $\mathbb{Z}$-form of $\mathfrak{U}(\mathfrak{g}_{l-1})^-$.

Set $\Pi' := \{K \in \Pi |\ k_j^i = 0,\ 1 < i \le l\}$.

3.4. If we define the ordinary vector addition in Π, one has the following claims:

(1) $\Pi = \Pi_{l-1} \oplus \Pi'$;

(2) If $K_2 \in \Pi_{l-1}$ and $K_1 \in \Pi'$, then $\theta^{K_2}\theta^{K_1} = \theta^{K_2+K_1}$;

(3) If $K_1, K_1' \in \Pi'$ with $K_1 \prec K_1'$, then $K_2 + K_1 \prec K_1'$ for any $K_2 \in \Pi_{l-1}$.

4. A new multiplicity formula of $V(\lambda)$

4.1. Let Λ be the set of weights for $\mathfrak{g}_l$, and $\omega_1, \omega_2, \cdots, \omega_l$ the set of fundamental dominant weights. Then the set of dominant weights Λ^+ is defined to be

$$\{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) = \sum_{i=1}^{l} \lambda_i \omega_i \ \text{ with all } \ \lambda_i \in \mathbb{N}\}.$$

Let E be the real vector space spanned by $\alpha_1, \alpha_2, \cdots, \alpha_l$. It is well-known that $\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_l^\vee$ again form a basis of E, and $\omega_1, \omega_2, \cdots, \omega_l$ form the dual basis relative to the inner product on E: $(\omega_i, \alpha_j^\vee) = \dfrac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{i\ j}$. If we restrict ourselves to considering the $(l-1)$-dimensional subspaces E' of E spanned by $\alpha_1, \alpha_2, \cdots, \alpha_{l-1}$, then $\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_{l-1}^\vee$ and $\omega_1, \omega_2, \cdots, \omega_{l-1}$ remain the dual bases of E' relative to the inner product on E. Therefore, we can consider the restriction of $\mathfrak{U}(\mathfrak{g}_l)$ to $\mathfrak{U}(\mathfrak{g}_{l-1})$, and the restriction of $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ as a weight of $\mathfrak{g}_l$ to $\lambda_{\mathfrak{g}_{l-1}} = (\lambda_1, \lambda_2, \cdots, \lambda_{l-1})$ as a weight of $\mathfrak{g}_{l-1}$. Moreover, let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \in \Lambda^+$ be a dominant weight, and v a maximal vector of weight λ of the $\mathfrak{U}(\mathfrak{g}_l)$-module $V(\lambda)$. Then $V(\lambda)|_{\mathfrak{U}(\mathfrak{g}_{l-1})}$ denotes the restriction of $V(\lambda)$ to a $\mathfrak{U}(\mathfrak{g}_{l-1})$-module.

We can make use of the recursive property of the basis $\{\theta^K |\ K \in \mathbb{N}^{\frac{l(l+1)}{2}}\}$ to construct a new basis of the finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ with $\lambda \in \Lambda^+$, and to get a new multiplicity formula of $V(\lambda)$. Following Littermann [2], we define λ_i^j in such a way: $\lambda_1^1 = \lambda_1$, and for $1 < j \le l$, λ_1^j is defined to be

$$\begin{aligned}
&h_1 \left(f_{l-j+2}^{(k_{l-j+2}^{j-1})} f_{l-j+1}^{(k_{l-j+1}^{j-1})} \cdots f_1^{(k_1^{j-1})} \cdots f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} v \right) \\
=&\lambda_1^j \left(f_{l-j+2}^{(k_{l-i+2}^{j-1})} f_{l-j+1}^{(k_{l-i+1}^{j-1})} \cdots f_1^{(k_1^{j-1})} \cdots f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} v \right) \\
=&\left(\lambda_1 + \sum_{q=1}^{j-1} k_2^q - 2\sum_{q=1}^{j-1} k_1^q \right) \left(f_{l-j+2}^{(k_{l-j+2}^{j-1})} f_{l-j+1}^{(k_{l-j+1}^{j-1})} \cdots f_1^{(k_1^{j-1})} \cdots f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} v \right);
\end{aligned}$$

for $i>1$ and $j=1$, λ_i^1 is defined to be

$$h_i\left(f_{i-1}^{(k_{i-1}^1)}\cdots f_1^{(k_1^1)}v\right)=\lambda_i^1\left(f_{i-1}^{(k_{i-1}^1)}\cdots f_1^{(k_1^1)}v\right)=(\lambda_i+k_{i-1}^1)\left(f_{i-1}^{(k_{i-1}^1)}\cdots f_1^{(k_1^1)}v\right);$$

for $i>1$ and $j>1$, λ_i^j is defined to be

$$h_i\left(f_{i-1}^{(k_{i-1}^j)}\cdots f_1^{(k_1^j)}\cdots f_l^{(k_l^1)}\cdots f_1^{(k_1^1)}v\right)=\lambda_i^j\left(f_{i-1}^{(k_{i-1}^j)}\cdots f_1^{(k_1^j)}\cdots f_l^{(k_l^1)}\cdots f_1^{(k_1^1)}v\right)$$
$$=\left(\lambda_i+\sum_{q=1}^{j}k_{i-1}^q+\sum_{q=1}^{j-1}k_{i+1}^q-2\sum_{q=1}^{j-1}k_i^q\right)\left(f_{i-1}^{(k_{i-1}^j)}\cdots f_1^{(k_1^j)}\cdots f_l^{(k_l^1)}\cdots f_1^{(k_1^1)}v\right).$$

Note that our definition is somewhat different from Littelmann's definition in [2 §7].

Then we define the following two index set which are related to λ (comparing with Littelmann's definition of $S(\lambda)$ in [2 §7]).

$$\Pi_\lambda=\Pi_{l,\lambda}:=\{K\in\Pi|\ 0\le k_i^j\le\lambda_i^j,\ 1\le i\le l,\ 1\le j\le l-i+1\}.$$

It is easy to see that Π_λ is a finite set. We shall show in Theorem 4.8 that the set $\{\theta^K v|\ K\in\Pi_\lambda\}$ forms a basis of the $\mathbb{Z}$-form of $V(\lambda)$.

For any $P\in\Pi'$, one has $P=(0,\cdots,0,p_l,\ p_{l-1},\cdots,p_1)$, if we set $p_0=0$, then we define

$$\Pi'_\lambda:=\{P\in\Pi'|\ p_i-p_{i-1}\le\lambda_i,\ 1\le i<l,\},$$

and set $\lambda-\sum_{i=1}^{l}p_i\alpha_i=\lambda-P\alpha$ for later use. We shall see in Theorem 4.7 that Π'_λ is also a finite set, and it becomes an index set of highest weights of irreducible composition factors of $V(\lambda)$ to be viewed as a $\mathfrak{g}_{l-1}$)-module.

4.2. Let V be a $\mathfrak{U}(\mathfrak{g}_l)$-module. we say a vector $v\in V$ to be a *primitive vector* of V, if there are two submodules V_1,V_2 with $V_2\subset V_1\subseteq V$ such that $v\in V_1$, $v\notin V_2$, and all e_i with $1\le i\le l$ vanish the canonical image of v in V_1/V_2.

Let V be a $\mathfrak{U}(\mathfrak{g}_l)$-module. According to [3], we can prove the following lemma similarly.

4.3. Lemma *Let w be a primitive vector of weight λ in V. Then V has a composition factor isomorphic to $V(\lambda)$.*

Furthermore, one has the following lemma (cf. [1 §21.4.]).

4.4. Lemma *Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)\in\Lambda^+$ be a dominant weight, and v a maximal vector of weight λ of $V(\lambda)$. Then one has*

$$f_i^{(\lambda_i+1)}v=0,\quad 1\le i\le l.$$

4.5. Lemma *Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)\in\Lambda^+$ be a dominant weight. Let V be a finite dimensional $\mathfrak{U}(\mathfrak{g}_l)$-module generated by a maximal vector v of weight λ of V. Then one has $V\simeq V(\lambda)$.*

Proof If V is an irreducible $\mathfrak{U}(\mathfrak{g}_l)$-module. Then $V\simeq V(\lambda)$. Otherwise, one has

$$V=V(\lambda)\oplus M$$

according to the completely reducibility, because V is a finite dimensional $\mathfrak{U}(\mathfrak{g}_l)$-module. But V is generated by a maximal vector, it must be an indecomposable $\mathfrak{U}(\mathfrak{g}_l)$-module. This is a contradiction.

4.6. Lemma *Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)\in\Lambda^+$ be a dominant weight, and $P=(p_l,p_{l-1},\cdots,p_1)$. Let $V(\lambda)$ be an irreducible $\mathfrak{U}(\mathfrak{g}_l)$-module with maximal vector v. If there is an i such that $p_i-p_{i-1}>\lambda_i\geq 0$. Then one has*

$$f_i^{(p_i)}f_{i-1}^{(p_{i-1})}\cdots f_1^{(p_1)}v=0.$$

Proof According to (2.2.7), one has

$$\begin{aligned}
&(f_i^{(p_i)}f_{i-1}^{(p_{i-1})})f_{i-2}^{(p_{i-2})}\cdots f_1^{(p_1)}v\\
=&\sum_{k=0}^{p_{i-1}} f_{i-1}^{(k)}\ {}_if_{i-1}^{(p_{i-1}-k)}f_i^{(p_i-k)}f_{i-2}^{(p_{i-2})}\cdots f_1^{(p_1)}v\\
=&\sum_{k=0}^{p_{i-1}} f_{i-1}^{(k)}\ {}_if_{i-1}^{(p_{i-1}-k)}f_{i-2}^{(p_{i-2})}\cdots f_1^{(p_1)}f_i^{(p_i-k)}v.
\end{aligned}$$

Note that $k\leq p_{i-1}$ and $0\leq\lambda_i<p_i-p_{i-1}\leq p_i-k$, the above summation is zero by lemma 4.4.

Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)\in\Lambda^+$ be a dominant weight. The finite-dimensional irreducible $\mathfrak{U}(\mathfrak{g}_l)$-module $V(\lambda)$ can be viewed as a $\mathfrak{U}(\mathfrak{g}_{l-1})$-module. It is no longer irreducible, and can be decomposed into a direct sum of irreducible $\mathfrak{U}(\mathfrak{g}_{l-1})$ -module. The following theorem tell us how one can decompose it.

4.7. Theorem *Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_l)\in\Lambda^+$ be a dominant weight. As a $\mathfrak{U}(\mathfrak{g}_{l-1})$-module, the irreducible $\mathfrak{U}(\mathfrak{g}_l)$-module $V(\lambda)$ has the following direct sum decomposition*

$$V(\lambda)|_{\mathfrak{U}(\mathfrak{g}_{l-1})}=\bigoplus_{P\in\Pi'_\lambda}V\left((\lambda-P\alpha)_{\mathfrak{g}_{l-1}}\right).$$

Proof By definition, Π'_λ is a finite set. Let $|\Pi'_\lambda|=t$. We can arrange elements of Π'_λ according to the ordering of Π'_λ defined in §1.2. Then one has

$$\Pi'_\lambda=\{\mathbf{0}=P_1\prec P_2\prec\cdots\prec P_t\}.$$

Set

$$M_{P_s}=\sum_{K\in\Pi,K\prec P_{s+1}}\mathbb{C}\theta^Kv,\quad 1\leq s\leq t-1,$$

where v is a maximal vector of $V(\lambda)$ and $M_{P_t}=V(\lambda)$. Then one has

$$0\subseteq M_{P_1}\subseteq M_{P_2}\subseteq\cdots\subseteq M_{P_t}=V(\lambda).$$

First of all, we show that M_{P_s}, $1\leq s\leq t$, is a $\mathfrak{U}(\mathfrak{g}_{l-1})$-submodule of $V(\lambda)$. It does so when $s=t$. We need only to consider cases of $1\leq s<t$. For any $\theta^Kv\in M_{P_s}$ with $K\prec P_{s+1}$, it is still a weight vector, and for any h_i with $1\leq i\leq l$, one has by (2.1.3)

$$h_i\theta^Kv=a_{i_K}\theta^Kv\in M_{P_s},\ \text{ with }\ a_{i_K}\in\mathbb{Z}.$$

By (3.4.1), $K = K_1 + K_2$ with $K_1 \in \Pi'$ and $K_2 \in \Pi_{l-1}$. Therefore, one has for any $f_i \in \mathfrak{U}(\mathfrak{g}_{l-1})$ with $1 \le i \le l-1$,

$$\begin{aligned} f_i\theta^K v =& f_i\theta^{K_1+K_2} v = f_i\left(\theta^{K_2}\theta^{K_1}\right) v \qquad\qquad \text{by (3.4.2)} \\ =& \left(f_i\theta^{K_2}\right)\theta^{K_1} v = \left(\sum_{K'\in\Pi_{l-1}} a_{_{K'}}\theta^{K'}\right)\theta^{K_1} v \\ =& \sum_{K'\in\Pi_{l-1}} a_{_{K'}}\theta^{K'+K_1} v, \ \text{ with } \ a_{K'} \in \mathbb{Z}. \end{aligned}$$

Note the fact that $K = K_1 + K_2 \prec P_{s+1}$, one has $K_1 \prec P_{s+1}$, and $K' + K_1 \prec P_{s+1}$ for any $K' \in \Pi_{l-1}$. Therefore,

$$f_i\theta^K v = \sum_{K'\in\Pi_{l-1}} a_{_{K'}}\theta^{K'+K_1} v \in M_{P_s}.$$

Furthermore, one has for any e_i with $1 \le i \le l$,

$$\begin{aligned} e_i\theta^K v =& e_i f_1^{(k_1^l)} f_2^{(k_2^{l-1})} f_1^{(k_1^{l-1})} \cdots f_i^{(k_i^{l-i+1})} f_{i-1}^{(k_{i-1}^{l-i+1})} \cdots f_1^{(k_1^{l-i+1})} \cdots \\ & f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} v \\ =& \theta^K e_i v + \sum_{n=1}^{l-i+1} f_1^{(k_1^l)} f_2^{(k_2^{l-1})} f_1^{(k_1^{l-1})} \cdots f_i^{(k_i^n-1)} (h_i - k_i^n + 1) \\ & f_{i-1}^{(k_{i-1}^n)} \cdots f_1^{(k_1^n)} \cdots f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} v \qquad\qquad \text{by (2.1.5)} \\ =& \sum_{n=1}^{l-i+1} f_1^{(k_1^l)} f_2^{(k_2^{l-1})} f_1^{(k_1^{l-1})} \cdots f_i^{(k_i^n-1)} f_{i-1}^{(k_{i-1}^n)} \cdots f_1^{(k_1^n)} \cdots \\ & f_l^{(k_l^1)} f_{l-1}^{(k_{l-1}^1)} \cdots f_1^{(k_1^1)} a_n v \qquad\qquad \text{by (2.1.3)} \\ =& \sum_{n=1}^{l-i+1} a_n\theta^{K-K_n} v, \end{aligned}$$

where

$$a_n = \lambda_i - k_i^n + 1 - 2\sum_{d=1}^{n-1} k_i^d + \sum_{d=1}^{n} k_{i-1}^d + \sum_{d=1}^{n-1} k_{i+1}^d \in \mathbb{Z},$$

and $K_n = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{N}^{\frac{l(l+1)}{2}}$ with 1 occurring in the place, where k_i^n lies in the corresponding K. Since $K - K_n \prec K \prec P_{s+1}$, one has

$$e_i\theta^K v = \sum_{n=1}^{l-i+1} a_n\theta^{K-K_n} v \in M_{P_s}.$$

It shows that M_{P_s} is stable under actions of e_i, h_i with $1 \le i \le l$ and f_i with $1 \le i \le l-1$, and M_{P_s} is a $\mathfrak{U}(\mathfrak{g}_{l-1})$-module.

Secondly, we show that $\theta^{P_s}v$, $1 \le s \le t$, are primitive vectors in $V(\lambda)$ when it is viewed as a $\mathfrak{U}(\mathfrak{g}_{l-1})$-module. Let $P_s = (0, \cdots 0, p_l, p_{l-1}, \cdots, p_1) \in \Pi'_\lambda$. Then one has

$$\begin{aligned}
& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} \theta^{P_s} v \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} v \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} \left(\sum_{k=0}^{p_l} f_l^{(p_l-k)} \binom{h_l - 2p_l + 2k}{k} e_l^{(p_l-k)} \right) f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} v \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} \binom{h_l}{p_l} f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} v \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \binom{h_l - p_{l-1}\alpha_{l-1}(h_l)}{p_l} f_{l-2}^{(p_{l-2})} \cdots f_1^{(p_1)} v \qquad \text{by (2.1.4)} \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} \binom{h_l - \sum_{k=1}^{l-1} p_k \alpha_k(h_l)}{p_l} v \qquad \text{by (2.1.4)} \\
=& e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} \binom{\lambda_l + p_{l-1}}{p_l} v = \cdots = \Pi_{k=1}^{l} \binom{\lambda_k + p_{k-1}}{p_k} v,
\end{aligned}$$

where $p_0 = 0$, the second equality is by (2.1.2), and the last third equality is because $\alpha_j(h_i) \neq 0$ if and only if $|i-j| \le 1$, and $\alpha_j(h_{j\pm1}) = -1, \alpha_j(h_j) = 2$. Note that $p_k - p_{k-1} \le \lambda_k$, one has $0 \le p_k \le \lambda_k + p_{k-1}$, and $\binom{\lambda_k + p_{k-1}}{p_k} \neq 0$ for all $1 \le k \le l$, i.e. $e_1^{(p_1)} \cdots e_{l-1}^{(p_{l-1})} e_l^{(p_l)} \theta^{P_s} v \neq 0$. This shows that $\theta^{P_s} v \neq 0$. By our construction, it is easy to see that $\theta^{P_s} v \in M_{P_s}$ but $\theta^{P_s} v \notin M_{P_{s-1}}$. Therefore, We need only to prove that $e_i \theta^{P_s} v \in M_{P_{s-1}}$ for $1 \le i \le l-1$, and then we can conclude that $\theta^{P_s} v$ is a primitive vector in $V(\lambda)$. In fact, one has for $1 \le i \le l$

$$\begin{aligned}
e_i \theta^{P_s} v =& e_i f_l^{(p_l)} f_{l-1}^{(p_{l-1})} \cdots f_1^{(p_1)} v \\
=& \theta^{P_s} e_i v + f_l^{(p_l)} \cdots f_i^{(p_i-1)} (h_i - p_i + 1) f_{i-1}^{(p_{i-1})} \cdots f_1^{(p_1)} v \qquad \text{by (2.1.5)} \\
=& (\lambda_i - p_i + 1 + p_{i-1}) f_l^{(p_l)} \cdots f_i^{(p_i-1)} f_{i-1}^{(p_{i-1})} \cdots f_1^{(p_1)} v \qquad \text{by (2.1.3)}
\end{aligned}$$

Since $(0, \cdots, 0, p_l, \cdots, p_{i+1}, p_i - 1, p_{i-1}, \cdots, p_1) \prec P_s$, one has $e_i \theta^{P_s} v \in M_{P_{s-1}}$ as required.

Thirdly, we show that $M_{P_s} = M_{P_{s-1}} + \mathfrak{U}(\mathfrak{g}_{l-1}) \theta^{P_s} v$. "$\supseteq$"is easy to be proved by definition of M_{P_s} and §3.4. Here we only prove "$\subseteq$". For any $K \in \Pi$ with $K \prec P_{s+1}$, one has a unique decomposition $K = K_2 + K_1$ with $K_2 \in \Pi'_\lambda$ and $K_1 \in \Pi_{l-1}$. If $K \prec P_s$, then $\theta^K v \in M_{P_{s-1}}$. Otherwise, when $P_s \preceq K \prec P_{s+1}$, we must have $K_2 = P_s$. Then

$$\theta^K v = \theta^{K_1 + K_2} v = \theta^{K_1} \theta^{P_s} v \in \mathfrak{U}(\mathfrak{g}_{l-1}) \theta^{P_s} v$$

as required.

Finally, we show that $M_{P_s}/M_{P_{s-1}} \simeq V\left((\lambda - P_s\alpha)_{\mathfrak{g}_{l-1}} \right)$. Let w be the canonical image of $\theta^{P_s} v$ in $M_{P_s}/M_{P_{s-1}}$. Then one has $M_{P_s}/M_{P_{s-1}} \simeq \mathfrak{U}(\mathfrak{g}_{l-1}) w$. Since $\theta^{P_s} v$ is a primitive vector in $V(\lambda)$, w becomes a maximal vector of weight $(\lambda - P_s\alpha)_{\mathfrak{g}_{l-1}}$. Note the fact that $V(\lambda)$ is a finite dimensional module, and $M_{P_s}/M_{P_{s-1}}$ is also finite dimensional

and generated by a maximal vector w, we must have $M_{P_s}/M_{P_{s-1}} \simeq V\left((\lambda - P_s\alpha)_{\mathfrak{g}_{l-1}}\right)$ by Lemma 4.5.

Using the complete reducibility, we complete the proof of Theorem 4.7.

The following theorem was proved in [2, Theorem 25].

4.8. Theorem *Let v be a maximal vector of $V(\lambda)$. Then $\{\theta^K v|\ K \in \Pi_\lambda\}$ forms a basis of the $\mathbb{Z}$-form of $V(\lambda)$.*

Proof We use induction on l. When $l = 1$, one has for any non-negative integer m that $\{f_1^{(i)} v|\ 0 \le i \le m\}$ forms a basis of the $\mathbb{Z}$-form of $V(m)$ by Lemma 4.4. Assume that our theorem holds for $l-1$, and then we have to show that the theorem holds for l. Let us use the same notations as in the proof of Theorem 4.7, and construct the bases of M_{P_s} for $1 \le s \le t$. For $s = 1$, one has $M_{P_1} \simeq V(\lambda_{\mathfrak{g}_{l-1}})$ as $\mathfrak{U}(\mathfrak{g}_{l-1})$-module, and $\{\theta^K v|\ K \in \Pi_{l-1,\lambda_{\mathfrak{g}_{l-1}}}\}$ is a basis of M_{P_1} by the induction hypothesis. When $s = 2$, note the following facts:

i) $\theta^{K+P_2} v \in M_{P_2}$ if $K \in \Pi_{l-1,(\lambda-P_2\alpha)_{\mathfrak{g}_{l-1}}}$ by §3.4(3);

ii) the number of $\{\theta^K | K \in \Pi_{l-1,(\lambda-P_2\alpha)_{\mathfrak{g}_{l-1}}}\}$ is equal to $\dim V\left((\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}\right)$ by the induction hypothesis;

iii) $M_{P_2}/M_{P_1} \simeq V\left((\lambda - P_2\alpha)_{\mathfrak{g}_{l-1}}\right)$.

Therefore, we see that

$$\{\theta^K v|\ K \in \Pi_{l-1,\lambda_{\mathfrak{g}_{l-1}}}\} \bigcup \{\theta^K \theta^{P_2} v = \theta^{K+P_2} v|\ K \in \Pi_{l-1,(\lambda-P_2\alpha)_{\mathfrak{g}_{l-1}}}\}$$

forms a basis of M_{P_2}.

In this way, the set of

$$\{\theta^K v|\ K \in \Pi_{l-1,\lambda_{\mathfrak{g}_{l-1}}}\} \bigcup \{\theta^{K+P_2} v|\ K \in \Pi_{l-1,(\lambda-P_2\alpha)_{\mathfrak{g}_{l-1}}}\} \bigcup \cdots \bigcup \{\theta^{K+P_t} v|\ K \in \Pi_{l-1,(\lambda-P_t\alpha)_{\mathfrak{g}_{l-1}}}\}$$

forms a basis of $M_{P_t} = V(\lambda)$. Note that elements in both the above set and the set of $\{\theta^K v|\ K \in \Pi_\lambda\}$ are same, this proves our theorem.

Denote by $\Pi(\lambda)$ the set of weights of the Weyl module $V(\lambda)$. Let $P = (0, \cdots, 0, p_l, p_{l-1}, \cdots, p_1) \in \Pi'_\lambda$. Then we say $P\alpha = \sum_{i=1}^l p_i\alpha_i \ll \sum_{i=1}^l a_i\alpha_i$ if and only if $p_l = a_l$ and $p_i \le a_i$ for all $i = 1, 2, \cdots, l-1$.

4.9. Theorem *Let $\mu \in \Pi(\lambda)$ be a weight of $V(\lambda)$. Then the multiplicity $m_\lambda(\mu)$ of μ in $V(\lambda)$ is equal to*

$$\begin{aligned} m_\lambda(\mu) = \dim V(\lambda)_\mu &= \sum_{P\in\Pi'_\lambda,\ P\alpha\ll\lambda-\mu} \dim V\left((\lambda - P\alpha)_{\mathfrak{g}_{l-1}}\right)_{(\mu_{\mathfrak{g}_{l-1}})} \\ &= \sum_{P\in\Pi'_\lambda,\ P\alpha\ll\lambda-\mu} m_{(\lambda-P\alpha)_{\mathfrak{g}_{l-1}}}\left(\mu_{\mathfrak{g}_{l-1}}\right). \end{aligned}$$

Proof Let us use the same notations as in the proof of Theorem 4.7, and let $\lambda - \mu = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_l\alpha_l$ with all $a_i \ge 0$, $i = 1, 2, \cdots, l$. Then the basis elements of weight μ in $V(\lambda)$ are $\mathcal{M} = \{\theta^K v|\ K = (k_1^l, k_2^{l-1}, k_1^{l-1}, \cdots, k_i^{l-i+1}, k_{i-1}^{l-i+1}, \cdots, k_1^{l-i+1}, \cdots, k_l^1, k_{l-1}^1, \cdots, k_1^1) \in \Pi_\lambda$ with $k_l^1 = a_l, k_{l-1}^1 + k_{l-1}^2 = a_{l-1}, \cdots, k_1^1 + k_2^1 + \cdots + k_l^1 = a_1\}$, and the number of $\mathcal{M}$ is equal to $m_\lambda(\mu)$. If we divide $\mathcal{M}$ into a disjoint union of $\mathcal{M}_i$, where $\mathcal{M}_i = \{\theta^K v|\ K \in \mathcal{M}$ with $P_i \prec K \prec P_{i+1}\}$. From Theorem 4.8, we see that $\mathcal{M}_i \subseteq M_{P_i}$, and the number of $\mathcal{M}_i$ is equal to $m_{(\lambda-P_i\alpha)}\left(\mu_{\mathfrak{g}_{l-1}}\right)$. Now Theorem 4.9 follows from Theorem 4.7.

5. Examples

5.1. When $l=2$, $\mathfrak{g}_l$ is of type A_2. One has for $\lambda = a\omega_1 + b\omega_2 = (a,b) \in \Lambda_+$ the following index sets:

$$\begin{aligned}
&\Pi = \{(k_1^2, k_2^1, k_1^1) |\ k_1^1 \le k_2^1\} \subseteq \mathbb{N}^3,\\
&\Pi' = \{(0, k_2^1, k_1^1) |\ k_1^1 \le k_2^1\} \subseteq \mathbb{N}^3,\\
&\Pi_\lambda = \{(k_1^2, k_2^1, k_1^1) |\ k_1^1 \le a, k_1^2 \le a + k_2^1 - 2k_1^1, k_2^1 \le b + k_1^1\} \subseteq \Pi,\\
&\Pi'_\lambda = \{(0, p_2, p_1) |\ p_1 \le a, p_2 - p_1 \le b\} \subseteq \Pi'.
\end{aligned}$$

In particular, if $\lambda = 2\omega_1 + 3\omega_2 = (2,3)$, then $\Pi_\lambda = \{(k_1^2, k_2^1, k_1^1) \in \Pi |\ k_1^1 \le 2, k_1^2 \le 2 + k_2^1 - 2k_1^1, k_2^1 \le 3 + k_1^1\}$, and $\Pi'_\lambda = \{P_1 = (0,0,0) \prec P_2 = (0,1,0) \prec P_3 = (0,2,0) \prec P_4 = (0,3,0) \prec P_5 = (0,1,1) \prec P_6 = (0,2,1) \prec P_7 = (0,3,1) \prec P_8 = (0,4,1) \prec P_9 = (0,2,2) \prec P_{10} = (0,3,2) \prec P_{11} = (0,4,2) \prec P_{12} = (0,5,2)\}$.

Moreover,
M_{P_1} has basis $\{v, f_1 v, f_1^{(2)} v\}$, and is isomorphic to $V(2)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_2}/M_{P_1} has basis $\{f_2 v, f_1 f_2 v, f_1^{(2)} f_2 v, f_1^{(3)} f_2 v\}$, and is isomorphic to $V(3)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_3}/M_{P_2} has basis $\{f_2^{(2)} v, f_1 f_2^{(2)} v, f_1^{(2)} f_2^{(2)} v, f_1^{(3)} f_2^{(2)} v, f_1^{(4)} f_2^{(2)}\}$, and is isomorphic to $V(4)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_4}/M_{P_3} has basis $\{f_2^{(3)} v, f_1 f_2^{(3)} v, f_1^{(2)} f_2^{(3)} v, f_1^{(3)} f_2^{(3)} v, f_1^{(4)} f_2^{(3)}, f_1^{(5)} f_2^{(3)} v\}$, and is isomorphic to $V(6)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_5}/M_{P_4} has basis $\{f_2 f_1 v, f_1 f_2 f_1 v\}$, and is isomorphic to $V(1)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_6}/M_{P_5} has basis $\{f_2^{(2)} f_1 v, f_1 f_2^{(2)} f_1 v, f_1^{(2)} f_2^{(2)} f_1 v\}$, and is isomorphic to $V(2)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_7}/M_{P_6} has basis $\{f_2^{(3)} f_1 v, f_1 f_2^{(3)} f_1 v, f_1^{(2)} f_2^{(3)} f_1 v, f_1^{(3)} f_2^{(3)} f_1 v\}$, and is isomorphic to $V(3)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_8}/M_{P_7} has basis $\{f_2^{(4)} f_1 v, f_1 f_2^{(4)} f_1 v, f_1^{(2)} f_2^{(4)} f_1 v, f_1^{(3)} f_2^{(4)} f_1 v, f_1^{(4)} f_2^{(4)} f_1 v\}$, and is isomorphic to $V(4)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
M_{P_9}/M_{P_8} has basis $\{f_2^{(2)} f_1^{(2)} v\}$, and is isomorphic to $V(0)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
$M_{P_{10}}/M_{P_9}$ has basis $\{f_2^{(3)} f_1^{(2)} v, f_1 f_2^{(3)} f_1^{(2)} v\}$, and is isomorphic to $V(1)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
$M_{P_{11}}/M_{P_{10}}$ has basis $\{f_2^{(4)} f_1^{(2)} v, f_1 f_2^{(4)} f_1^{(2)} v, f_1^{(2)} f_2^{(4)} f_1^{(2)} v\}$, and is isomorphic to $V(2)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules;
$M_{P_{12}}/M_{P_{11}}$ has basis $\{f_2^{(5)} f_1^{(2)} v, f_1 f_2^{(5)} f_1^{(2)} v, f_1^{(2)} f_2^{(5)} f_1^{(2)} v, f_1^{(3)} f_2^{(5)} f_1^{(2)} v\}$, and is isomorphic to $V(3)$ as $\mathfrak{U}(\mathfrak{g}_1)$-modules.

Put all these elements together, we get a basis of $V(2\omega_1 + 3\omega_2)$. Furthermore, one has $V(2\omega_1 + 3\omega_2)|_{\mathfrak{U}(\mathfrak{g}_1)} \simeq \bigoplus_{i=1}^{12} V(\lambda - P_i\alpha)|_{\mathfrak{U}(\mathfrak{g}_1)}$. It is known that $m_\lambda(\mu) = 3$ for $\mu = \omega_2$, and $\lambda - \mu = 2\alpha_1 + 2\alpha_2$. Using Theorem 4.9, one has $m_\lambda(\mu) = m_{(\lambda - P_3\alpha)_{\mathfrak{g}_1}}(\mu_{\mathfrak{g}_1}) + m_{(\lambda - P_6\alpha)_{\mathfrak{g}_1}}(\mu_{\mathfrak{g}_1}) + m_{(\lambda - P_9\alpha)_{\mathfrak{g}_1}}(\mu_{\mathfrak{g}_1}) = m_4(0) + m_2(0) + m_0(0) = 1 + 1 + 1 = 3$.

5.2. When $l=4$, $\mathfrak{g}_l$ is of type A_4. One has for $\lambda = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4 = (a,b,c,d) \in \Lambda_+$ the following index sets:

$$\begin{aligned}
\Pi &= \{(k_1^4, k_2^3, k_1^3, k_3^2, k_2^2, k_1^2, k_4^1, k_3^1, k_2^1, k_1^1) |\ k_1^3 \le k_2^3, k_1^2 \le k_2^2 \le k_3^2, k_1^1 \le k_2^1 \le k_3^1 \le k_4^1\} \\
&\quad \subseteq \mathbb{N}^{10}, \\
\Pi' &= \{(0, \cdots, 0, k_4^1, k_3^1, k_2^1, k_1^1) |\ k_1^1 \le k_2^1 \le k_3^1 \le k_4^1\} \subseteq \mathbb{N}^{10}, \\
\Pi_\lambda &= \{(k_1^4, k_2^3, k_1^3, k_3^2, k_2^2, k_1^2, k_4^1, k_3^1, k_2^1, k_1^1) |\ k_1^4 \le a + k_2^3 + k_2^2 + k_2^1 - 2k_1^3 - 2k_1^2 - 2k_1^1, \\
&\qquad k_2^3 \le b + k_1^3 + k_1^2 + k_1^1 + k_3^2 + k_3^1 - 2k_2^2 - 2k_2^1, k_1^3 \le a + k_2^2 + k_2^1 - 2k_1^2 - 2k_1^1, \\
&\qquad k_3^2 \le c + k_2^2 + k_2^1 + k_4^1 - 2k_3^1, k_2^2 \le b + k_1^2 + k_1^1 + k_1^3 - 2k_2^1, k_1^2 \le a + k_2^1 - 2k_1^1, \\
&\qquad k_4^1 \le d + k_3^1, k_3^1 \le c + k_2^1, k_2^1 \le b + k_1^1, k_1^1 \le a\} \subseteq \Pi, \\
\Pi'_\lambda &= \{(0, \cdots, 0, p_4, p_3, p_2, p_1) |\ p_1 \le a, p_2 - p_1 \le b, p_3 - p_2 \le c, p_4 - p_3 \le d\} \subseteq \Pi'.
\end{aligned}$$

If we take $\lambda = \omega_1 + \omega_2 + \omega_3 + \omega_4 = (1,1,1,1)$, then $\Pi_\lambda = \{(k_1^4, k_2^3, k_1^3, k_3^2, k_2^2, k_1^2, k_4^1, k_3^1, k_2^1, k_1^1) \in \Pi |\ k_1^4 \le 1 + k_2^3 + k_2^2 + k_2^1 - 2k_1^3 - 2k_1^2 - 2k_1^1, k_2^3 \le 1 + k_1^3 + k_1^2 + k_1^1 + k_3^2 + k_3^1 - 2k_2^2 - 2k_2^1, k_1^3 \le 1 + k_2^2 + k_2^1 - 2k_1^2 - 2k_1^1, k_3^2 \le 1 + k_2^2 + k_2^1 + k_4^1 - 2k_3^1, k_2^2 \le 1 + k_1^2 + k_1^1 + k_1^3 - 2k_2^1, k_1^2 \le 1 + k_2^1 - 2k_1^1, k_4^1 \le 1 + k_3^1, k_3^1 \le 1 + k_2^1, k_2^1 \le 1 + k_1^1, k_1^1 \le 1\}$, and $\Pi'_\lambda = \{P_1 = (0, \cdots, 0) \prec P_2 = (0, \cdots, 0, 1, 0, 0, 0) \prec P_3 = (0, \cdots, 0, 1, 1, 0, 0) \prec P_4 = (0, \cdots, 2, 1, 0, 0) \prec P_5 = (0, \cdots, 0, 1, 1, 1, 0) \prec P_6 = (0, \cdots, 0, 2, 1, 1, 0) \prec P_7 = (0, \cdots, 0, 2, 2, 1, 0) \prec P_8 = (0, \cdots, 0, 3, 2, 1, 0) \prec P_9 = (0, \cdots, 0, 1, 1, 1, 1) \prec P_{10} = (0, \cdots, 0, 2, 1, 1, 1) \prec P_{11} = (0, \cdots, 0, 2, 2, 1, 1) \prec P_{12} = (0, \cdots, 0, 3, 2, 1, 1) \prec P_{13} = (0, \cdots, 0, 2, 2, 2, 1) \prec P_{14} = (0, \cdots, 0, 3, 2, 2, 1) \prec P_{15} = (0, \cdots, 0, 3, 3, 2, 1) \prec P_{16} = (0, \cdots, 0, 4, 3, 2, 1)\}$.

Therefore, one has the following isomorphisms of $\mathfrak{U}(\mathfrak{g}_3)$-modules:

$$\begin{aligned}
&M_{P_1} \simeq V(1,1,1), && M_{P_2}/M_{P_1} \simeq V(1,1,2), && M_{P_3}/M_{P_2} \simeq V(1,2,0), \\
&M_{P_4}/M_{P_3} \simeq V(1,2,1), && M_{P_5}/M_{P_4} \simeq V(2,0,1), && M_{P_6}/M_{P_5} \simeq V(2,0,2), \\
&M_{P_7}/M_{P_6} \simeq V(2,1,0), && M_{P_8}/M_{P_7} \simeq V(2,1,1), && M_{P_9}/M_{P_8} \simeq V(0,1,1), \\
&M_{P_{10}}/M_{P_9} \simeq V(0,1,2), && M_{P_{11}}/M_{P_{10}} \simeq V(0,2,0), && M_{P_{12}}/M_{P_{11}} \simeq V(0,2,1), \\
&M_{P_{13}}/M_{P_{12}} \simeq V(1,0,1), && M_{P_{14}}/M_{P_{13}} \simeq V(1,0,2), && M_{P_{15}}/M_{P_{14}} \simeq V(1,1,0), \\
&M_{P_{16}}/M_{P_{15}} \simeq V(1,1,1).
\end{aligned}$$

Moreover, one has $V(\omega_1+\omega_2+\omega_3+\omega_4)|_{\mathfrak{U}(\mathfrak{g}_3)} \simeq \bigoplus_{i=1}^{16} V(\lambda - P_i\alpha)|_{\mathfrak{U}(\mathfrak{g}_3)}$, and $m_\lambda(\mu) = 8$ for $\mu = \omega_2 + \omega_3 = (0,1,1,0)$ with $\lambda - \mu = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Using Theorem 4.9, one has

$$\begin{aligned}
m_\lambda(\mu) =& m_{(\lambda - P_2\alpha)_{\mathfrak{g}_3}}(\mu_{\mathfrak{g}_3}) + m_{(\lambda - P_3\alpha)_{\mathfrak{g}_3}}(\mu_{\mathfrak{g}_3}) + m_{(\lambda - P_5\alpha)_{\mathfrak{g}_3}}(\mu_{\mathfrak{g}_3}) + m_{(\lambda - P_9\alpha)_{\mathfrak{g}_3}}(\mu_{\mathfrak{g}_3}) \\
=& m_{(1,1,2)}(0,1,1) + m_{(1,2,0)}(0,1,1) + m_{(2,0,1)}(0,1,1) + m_{(0,1,1)}(0,1,1) \\
=& 4 + 2 + 1 + 1 = 8.
\end{aligned}$$

Acknowledgement

This work is supported in part by the National Natural Science Foundation of China (10271088). The first named author is also grateful to the Abdus Salam International Centre for Theoretical Physics for its financial support and hospitability during his visit.

References

[1] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory,* GTM 9, Springer-Verlag, New York/Heidelberg/Berlin, 1972.

[2] Littelmann, P., *An algorithm to compute bases and representation matrices for SL_{n+1}-representations*, J. Pure and Appl. Algebra **117&118** (1997), 447–468.

[3] Xi, Nanhua, *Maximal and Primitive elements in Weyl modules for type A_2*, J. Algebra **215(2)** (1999), 735–756.

[4] Xu, Baoxing and Ye, Jiachen, *Irreducible characters of algebraic groups in characteristic two* (I), Algebra Colloquium **4(3)** (1997), 281–290.

[5] Ye, Jiachen and Zhou, Zhongguo, *Irreducible characters of algebraic groups in characteristic two* (III), Commun. Algebra **28(9)** (2000), 4227-4247.